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BOUNDING PRINCIPLES FOR TWO-PHASE FLOW SYSTEMS

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Abstract—For two classes of multiphase flow problems, upper and lower bounding principles are constructed for the rate of dissipation of mechanical energy as the result of viscous forces both in the bulk fluid and in the phase interface. These principles are developed for simple classes of non-linear constitutive equations for the bulk stress tensor and for the surface stress tensor. The integral mechanical energy balance relates these bounds to quantities that are subject to direct experimental evaluation.

INTRODUCTION

The purpose here is to develop bounding principles for two-phase flow systems that take into account the effect of the interfacial stress which acts in the phase interface separating the two phases. These bounding principles are of practical value because of the difficulty in obtaining exact solutions for this type of problem.

The bounding principles developed in this paper are based upon those given by Hill (1956) and by Hill & Power (1956) (for a more complete development, see Ehrlich & Slattery 1968 or Slattery 1972; it is also interesting to look at a later development from a different point of view by Johnson 1960, 1961). Hill's bounding principles have been used previously to analyze single-phase flow systems (see for example Ehrlich & Slattery 1968 or Hopke & Slattery 1970a, 1970b).

We begin with a brief summary of the results from Hill's work that we shall be using here. This is followed by an extension of Hill's bounding principles to the flow of two phases and a two-dimensional analog of Hill's bounding principles. We conclude with an indication of how these bounding principles are to be applied in practice.

RESULTS FROM HILL

Hill restricted his attention to incompressible fluids whose stress-deformation behavior could be represented by the generalized Newtonian model:

$$\mathbf{S} \equiv \mathbf{T} + p\mathbf{I} = 2\eta\mathbf{D} \qquad [1]$$

$$\eta = \eta(\mathbf{D}). \qquad [2]$$

Here $\mathbf{S}$ is the extra stress tensor, $\mathbf{T}$ is the stress tensor, p is the mean pressure, $\mathbf{I}$ is the identity tensor, η is the apparent shear viscosity function, $\mathbf{D}$ is the rate of deformation tensor,

$$\mathbf{D} \equiv \tfrac{1}{2}\,[\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}] \qquad [3]$$

and

$$D \equiv [\operatorname{tr}(\mathbf{D}\cdot\mathbf{D})]^{1/2}. \quad [4]$$

For this model of material behavior, there are two scalar potential functions

$$E = E(D) \equiv \int_0^{D^2} \eta \, \mathrm{d}D^2 \quad [5]$$

and

$$E_c = E_c(S) \equiv \int_0^{S^2} \frac{1}{4\eta} \, \mathrm{d}S^2 \quad [6]$$

such that

$$\mathbf{S} = \frac{\partial E}{\partial \mathbf{D}} \quad [7]$$

and

$$\mathbf{D} = \frac{\partial E_c}{\partial \mathbf{S}}. \quad [8]$$

In [6] we have defined

$$S \equiv [\operatorname{tr}(\mathbf{S}\cdot\mathbf{S})]^{1/2}. \quad [9]$$

For the definition of the gradient of a scalar function with respect to a second-order tensor, see Slattery (1972) or Truesdell & Noll (1965).

Hill's primary results follow from the realization that both of these potential functions can be required to be convex:

$$E(D^*) - E(D) \geq \operatorname{tr}\left[\frac{\partial E}{\partial \mathbf{D}}\cdot(\mathbf{D}^* - \mathbf{D})\right] \quad [10]$$

and

$$E_c(S^*) - E_c(S) \geq \operatorname{tr}\left[\frac{\partial E^c}{\partial \mathbf{S}}\cdot(\mathbf{S}^* - \mathbf{S})\right]. \quad [11]$$

Sufficient conditions for the convexity of these potential functions are

$$2\eta = \frac{1}{D}\frac{\mathrm{d}E}{\mathrm{d}D} \geq 0 \quad [12]$$

and

$$\frac{\mathrm{d}S}{\mathrm{d}D} = \frac{\mathrm{d}^2E}{\mathrm{d}D^2} \geq 0. \quad [13]$$

These conditions are consistent with observed fluid behavior. These two potential functions are related by

$$E + E_c = \operatorname{tr}(\mathbf{S}\cdot\mathbf{D}) \quad [14]$$

which we will take advantage of in writing [10] and [11] as upper and lower bounds on E.

Finally, so long as E is a homogeneous function of degree r (Kaplan 1952),

$$rE = \operatorname{tr}(\mathbf{S} \cdot \mathbf{D}). \tag{15}$$

This relationship will allow us to relate bounds for the potential function E to a quantity that has more direct physical significance.

APPLICATION TO TWO-PHASE FLOW

Prompted by Hill, let us see what the convexity of E and of E_c imply about a system containing two phases. We will make the following assumptions about this system.

(1) Each phase is an incompressible, generalized Newtonian fluid. Consequently the equation of continuity reduces to

$$\operatorname{div} \mathbf{v} = 0, \tag{16}$$

where $\mathbf{v}$ is the velocity vector and div denotes the usual divergence operation.

(2) The external force per unit mass $\mathbf{f}$ (gravity) may be represented by the gradient of a scalar potential φ (potential energy per unit mass),

$$\mathbf{f} = -\nabla\varphi. \tag{17}$$

(3) The velocity distribution is independent of time. The speed of displacement of the phase interface (the normal component of the velocity of a point on the surface) is consequently zero.

(4) Inertial effects are negligible with respect to viscous effects and the equation of motion for each phase reduces to

$$\operatorname{div}(\mathbf{T} - \rho\varphi\mathbf{I}) = 0. \tag{18}$$

Here ρ is the mass density.

(5) There is no mass transfer across the phase interface. As a result, the normal component of velocity at the interface is zero.

(6) The tangential components of velocity are continuous across the interface.

(7) Any surfactant present in the interface is uniformly distributed over the entire surface.

The essential aspects of the general two-phase system are shown in figure 1. The phase interface Σ separates the region R^+ occupied by the plus phase from the region R^- occupied by the minus phase. The total region occupied by the system is denoted by R,

$$R = R^+ + R^-. \tag{19}$$

Since the location of the phase interface is not always a known *a priori*, we will find it necessary at one stage in our discussion to assume that its location is Σ^*. This assumed phase interface location subdivides the regions actually occupied by the two phases:

$$R^+ = R^{++} + R^{+-} \tag{20}$$

$$R^- = R^{--} + R^{-+}. \tag{21}$$

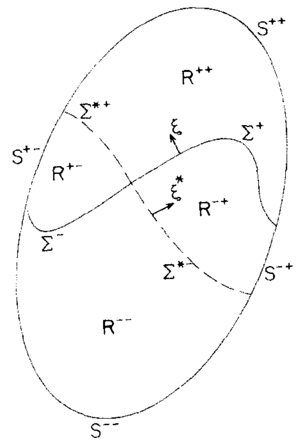

Figure 1. Actual phase interface, $\Sigma = \Sigma^+ + \Sigma^-$, and trial interface, $\Sigma^* = \Sigma^{*+} + \Sigma^{*-}$.

Here R^{++} is that portion of R^+ on the plus side of both Σ and Σ^*; R^{+-} is that portion of R^+ on the minus side of Σ^*; R^{--} is that portion of R^- on the minus side of both Σ and Σ^*; R^{-+} is that portion of R^- on the plus side of Σ^*. The actual phase interface Σ and the assumed phase interface Σ^* are split as well:

$$\Sigma = \Sigma^+ + \Sigma^- \qquad [22]$$

$$\Sigma^* = \Sigma^{*+} + \Sigma^{*-}. \qquad [23]$$

In these equations, Σ^+ is that portion of Σ on the plus side of Σ^*; Σ^- is that portion of Σ on the minus side of Σ^*; Σ^{*+} is that portion of Σ^* on the plus side of Σ; Σ^{*-} is that portion of Σ^* on the minus side of Σ. Finally, the closed bounding surface S of the two-phase system is composed of four regions,

$$S = S^{++} + S^{+-} + S^{-+} + S^{--}, \qquad [24]$$

where S^{++} is that portion of S on the plus side of both Σ and Σ^*; S^{+-} is that portion of S on the plus side of Σ and the minus side of Σ^*; S^{-+} is that portion of S on the minus side of Σ and the plus side of Σ^*; S^{--} is that portion of S on the minus side of both Σ and Σ^*. In [19]–[21], we are anticipating several applications of Green's transformation; our object has been to split R into four regions in which the actual density, velocity, stress, . . . distributions as well as their approximations are continuous functions of position. Equations [22]–[24] provide us with the notation for the closed bounding surfaces of these sub-regions.

In what follows, an asterisk* denotes an approximation for a distribution or configuration. The nature of the approximation will be developed as we proceed.

Recognizing [7], let us integrate [10] over both phases to find

$$\int_R \{E(D^*) - E(D) - \text{tr}\,[\mathbf{S}\cdot(\mathbf{D}^* - \mathbf{D})]\}\,dV \geq 0, \qquad [25]$$

where tr denotes the trace of a second-order tensor and $\mathrm{d}V$ indicates that a volume integration is to be performed. In this inequality, we regard D^* and $\mathbf{D}^*$ as being defined in terms of a trial or approximate velocity distribution $\mathbf{v}^*$ that satisfies the equation of continuity, continuity of velocity at Σ^*, and all of the boundary conditions on the actual velocity distribution,

$$\text{on} \quad S_v : \mathbf{v}^* = \mathbf{v}. \tag{26}$$

Here and in what follows we denote by S_v that portion of S on which velocity is specified. We can write the integral for the rate of production of internal energy in the region R^{++} as

$$\begin{aligned}\int_{R^{++}} \operatorname{tr}(\mathbf{S}\cdot\mathbf{D})\,\mathrm{d}V &= \int_{R^{++}} \operatorname{tr}\,[(\mathbf{T} - \rho\varphi\mathbf{I})\cdot\nabla\mathbf{v}]\,\mathrm{d}V \\ &= \int_{S^{++}} \mathbf{v}\cdot(\mathbf{T} - \rho\varphi\mathbf{I})\cdot\mathbf{n}\,\mathrm{d}A \\ &\quad - \int_{\Sigma^{+}} \mathbf{v}\cdot(\mathbf{T} - \rho\varphi\mathbf{I})\cdot\boldsymbol{\xi}^{+}\,\mathrm{d}A \\ &\quad - \int_{\Sigma^{*+}} \mathbf{v}\cdot(\mathbf{T} - \rho\varphi\mathbf{I})\cdot\boldsymbol{\xi}^{*+}\,\mathrm{d}A.\end{aligned} \tag{27}$$

Here $\mathbf{n}$ is the outwardly directed unit normal to the closed suface S: $\boldsymbol{\xi}^{+}$ is the unit normal to Σ directed into the plus phase; $\boldsymbol{\xi}^{*+}$ is the unit normal to Σ^* directed into the plus phase; $\mathrm{d}A$ indicates that an area integration is to be performed. In arriving at this result, we have placed no restrictions on Σ^*, the approximation to the actual phase interface configuration. If we add [27] and the similar relationships for R^{+-}, R^{-+}, and R^{--}, we find

$$\begin{aligned}\int_{R} \operatorname{tr}(\mathbf{S}\cdot\mathbf{D})\,\mathrm{d}V &= \int_{S} \mathbf{v}\cdot(\mathbf{T} - \rho\varphi\mathbf{I})\cdot\mathbf{n}\,\mathrm{d}A \\ &\quad - \int_{\Sigma} \mathbf{v}\cdot[\mathbf{T}\cdot\boldsymbol{\xi} - \rho\varphi\boldsymbol{\xi}]\,\mathrm{d}A\end{aligned} \tag{28}$$

where the boldface brackets indicate the jump of the quantity enclosed across the interface:

$$[\beta\boldsymbol{\xi}] \equiv \beta^{+}\boldsymbol{\xi}^{+} + \beta^{-}\boldsymbol{\xi}^{-} = (\beta^{+} - \beta^{-})\boldsymbol{\xi}^{+}. \tag{29}$$

Using this result, we can write [25] as

$$\begin{aligned}\int_{R} E(D)\,\mathrm{d}V \leq \int_{R} E(D^*)\,\mathrm{d}V &+ \int_{S-S_v} (\mathbf{v} - \mathbf{v}^*)\cdot(\mathbf{T} - \rho\varphi\mathbf{I})\cdot\mathbf{n}\,\mathrm{d}A \\ &- \int_{\Sigma} (\mathbf{v} - \mathbf{v}^*)\cdot[\mathbf{T}\cdot\boldsymbol{\xi} - \rho\varphi\boldsymbol{\xi}]\,\mathrm{d}A.\end{aligned} \tag{30}$$

This will be known as the velocity extremum principle; it provides us with an upper bound on the volume integral of E, if the three integrals on the right side can be evaluated.

Let us now employ [8] in integrating [11] over both phases:

$$\int_R \{E_c(S^*) - E_c(S) - \operatorname{tr}[\mathbf{D}\cdot(\mathbf{S}^* - \mathbf{S})]\}\,\mathrm{d}V \geq 0. \qquad [31]$$

We define S^* and $\mathbf{S}^*$ in terms of a trial or approximate stress distribution $\mathbf{T}^*$ that satisfies the equation of motion [18] as well as the jump momentum balance at the phase interface. In order to be consistent with [1], we require

$$\mathbf{S}^* \equiv \mathbf{T}^* - \tfrac{1}{3}\operatorname{tr}\mathbf{T}^*\mathbf{I}. \qquad [32]$$

Using [14] and [28], we are finally able to arrange [31] as

$$\int_R E(D)\,\mathrm{d}V \geq -\int_R E_c(S^*)\,\mathrm{d}V + \int_S \mathbf{v}\cdot(\mathbf{T}^* - \rho^*\varphi\mathbf{I})\cdot\mathbf{n}\,\mathrm{d}A$$
$$- \int_{\Sigma^*} \mathbf{v}\cdot[\mathbf{T}^*\cdot\boldsymbol{\xi}^* - \rho^*\varphi\boldsymbol{\xi}^*]\,\mathrm{d}A \qquad [33]$$

where ρ^* denotes the density distribution consistent with Σ^*. In arriving at this result, we have placed no restrictions on the trial interface configuration Σ^*. This gives us a lower bound for the volume integral of the potential function E. We will refer to [33] as the stress extremum principle.

The last integrals on the right sides of [30] and [33] distinguish the velocity and stress extremum principles obtained here from those proposed by Hill. It is also these integrals that make [30] and [33] impractical. In general we know neither the actual configuration of the phase interface nor the velocity and stress distributions at the phase interface.

UNIFORM SURFACE TENSION

In this section we will neglect all interfacial effects other than a uniform surface tension. We must place two additional restrictions upon the class of physical problems to which the results of this section will be applicable.

(8′) We must be willing to assume that the actual phase interface Σ belongs to a given family of parallel surfaces, from which the trial surface Σ^* will be chosen. For example, we may have experimental evidence to suggest that the interface is a plane parallel to the plane $z = 0$ in rectangular Cartesian coordinates.

(9′) We must also say that the actual velocity distribution $\mathbf{v}$ is everywhere tangent to this family of parallel surfaces. More explicitly, not only must $\mathbf{v}$ be tangent to Σ and Σ^*, but we must also choose $\mathbf{v}^*$ in such a manner that it will be everywhere tangent to Σ^*.

These assumptions may be satisfied at least approximately for flows that appear to be unidirectional.

If we neglect all interfacial effects other than a uniform surface tension and if we assume that there is no mass transfer across the phase interface, the jump mass balance is satisfied identically and the jump momentum balance reduces to (Slattery 1972)

$$\text{at } \Sigma:\quad [\mathbf{T}\cdot\boldsymbol{\xi}] = -2H\gamma\boldsymbol{\xi}. \qquad [34]$$

Here H is the mean curvature (McConnell 1957) and γ is the surface tension. We will require that the trial stress distribution $\mathbf{T}^*$ satisfies the same form of relationship at Σ^*.

Under these circumstances, [30] and [33] reduce to

$$\int_R E(D)\,\mathrm{d}V \leq \int_R E(D^*)\,\mathrm{d}V + \int_{S-S_v} (\mathbf{v} - \mathbf{v}^*)\cdot(\mathbf{T} - \rho\varphi\mathbf{I})\cdot\mathbf{n}\,\mathrm{d}A \qquad [35]$$

and

$$\int_R E(D)\,\mathrm{d}V \geq -\int_R E_c(S^*)\,\mathrm{d}V + \int_S \mathbf{v}\cdot(\mathbf{T}^* - \rho^*\varphi\mathbf{I})\cdot\mathbf{n}\,\mathrm{d}A. \qquad [36]$$

To review, the trial velocity distribution $\mathbf{v}^*$ must satisfy the equation of continuity and all of the boundary conditions on the actual velocity distribution; it must be both continuous at Σ^* and everywhere tangent to Σ^*. The trial stress distribution $\mathbf{T}^*$ must be symmetric and it must satisfy both the equation of motion [18] and the jump momentum balance [34]; $\mathbf{S}^*$ is defined in terms of $\mathbf{T}^*$ by [32]. The actual velocity distribution $\mathbf{v}$ is assumed to be tangent to both Σ and Σ^*.

The steady-state, stratified, laminar flow of two incompressible fluids through a duct is a flow for which [35] and [36] apply, at least approximately. We cannot be more definite, since the validity of assumptions (8′) and (9′) cannot be checked without both an exact solution and an analysis of its stability. For homogeneous potential functions E of order r as described by [15], the integral mechanical energy balance (Slattery 1972) reduces to

$$\Delta pQ/L = \frac{1}{L}\int_V \mathrm{tr}\,(\mathbf{S}\cdot\mathbf{D})\,\mathrm{d}V = \frac{r}{L}\int_V E\,\mathrm{d}V. \qquad [37]$$

Here Δp is the pressure drop in a duct of length L; Q is the total flow rate of both phases through the duct. Two approaches are possible.

(i) Given the ratio of the volume flow rates as well as the pressure drop, we can use [35] to obtain an upper bound to the total volume flow rate. In this calculation we obtain an approximate interface configuration Σ^*, which can be employed together with [36] to find an approximate lower bound for Q. This lower bound is approximate, since Σ^* is not directly fixed by the ratio of the volume flow rates. It is a function of how accurately the trial velocity distribution represents the actual one.

(ii) If we are given the holdup of each phase in the duct as well as the pressure drop, Σ is defined and [35] and [36] give upper and lower bounds for Q.

MORE GENERAL INTERFACIAL STRESSES

In addition to the seven listed previously, we must place another restriction upon the class of physical problems to which the results of this section will be applicable.

(8″) The configuration and location of the phase interface must be known *a priori*. For example, the interface may be a horizontal plane, the elevation of which is not a function of the flow. This means that the normal component of the trial velocity distribution $\mathbf{v}^*$ will be required to be zero at Σ.

For these problems, the last terms on the right of [30] and [33] rearrange into more readily useable forms for a broad class of interfacial behaviors.

We have already restricted ourselves to steady-state flows in which mass transfer across the phase interface can be neglected and in which any surfactant present in the interface is uniformly distributed over the surface. If we now recognize that mass may be associated with the interface, the jump mass balance reduces under these conditions to (Scriven 1960; Slattery 1967).

$$\operatorname{div}_{(\sigma)}\mathbf{v}^{(\sigma)} = \operatorname{div}_{(\sigma)}\mathbf{v} = 0, \qquad [38]$$

where $\mathbf{v}^{(\sigma)}$ is the surface velocity vector. The tangential components of $\mathbf{v}^{(\sigma)}$ are assumed to be equal to the tangential components of $\mathbf{v}$ in either adjoining phase evaluated at Σ; the normal component is the speed of displacement of the interface. Since there is assumed to be no mass transfer across the phase interface, we have identified at Σ

$$\mathbf{v}^{(\sigma)} = \mathbf{v}. \qquad [39]$$

The surface divergence operator $\operatorname{div}_{(\sigma)}$ is discussed in the appendix.

When interfacial stresses are taken into account, the jump momentum balance becomes (Slattery 1964, 1967)

$$[\mathbf{T} \cdot \boldsymbol{\xi}] = -\operatorname{div}_{(\sigma)}\mathbf{T}^{(\sigma)}, \qquad [40]$$

in which $\mathbf{T}^{(\sigma)}$ is the surface stress tensor (see appendix). This form of the jump momentum balance neglects the effects of inertial and external forces within the interface as well as mass transfer across the interface.

We will assume here that the surface stress tensor $\mathbf{T}^{(\sigma)}$ is a function of the surface rate of deformation tensor (see appendix)

$$\begin{aligned} \mathbf{D}^{(\sigma)} &\equiv \tfrac{1}{2}[\mathbf{P} \cdot \nabla_{(\sigma)}\mathbf{v}^{(\sigma)} + (\nabla_{(\sigma)}\mathbf{v}^{(\sigma)})^{T} \cdot \mathbf{P}] \\ &= \tfrac{1}{2}[\mathbf{P} \cdot \nabla_{(\sigma)}\mathbf{v} + (\nabla_{(\sigma)}\mathbf{v})^{T} \cdot \mathbf{P}]. \end{aligned} \qquad [41]$$

We have introduced here the projection tensor $\mathbf{P}$ that transforms every vector on the surface into its tangential component (see appendix); the surface gradient operator $\nabla_{(\sigma)}$ is defined in the appendix; the superscript T denotes the transpose of a second-order tensor. The alternative expression for $\mathbf{D}^{(\sigma)}$ follows from [39]. The most general function of this form that also satisfies the principle of material frame-indifference (Truesdell & Noll 1965; Slattery 1972) is

$$\mathbf{T}^{(\sigma)} = (\gamma + \lambda)\mathbf{P} + 2\varepsilon\mathbf{D}^{(\sigma)}, \qquad [42]$$

where

$$\lambda = \lambda(\operatorname{tr}\mathbf{D}^{(\sigma)}, D^{(\sigma)}), \qquad \varepsilon = \varepsilon(\operatorname{tr}\mathbf{D}^{(\sigma)}, D^{(\sigma)}) \qquad [43]$$

$$D^{(\sigma)} \equiv [\operatorname{tr}(\mathbf{D}^{(\sigma)} \cdot \mathbf{D}^{(\sigma)})]^{1/2}. \qquad [44]$$

In view of [38]

$$\operatorname{tr}\mathbf{D}^{(\sigma)} = \operatorname{div}_{(\sigma)}\mathbf{v}^{(\sigma)} = 0, \qquad [45]$$

and we will assume that

$$\lambda(0, D^{(\sigma)}) = 0. \qquad [46]$$

For this description of material behavior, there are two scalar potential functions

$$E^{(\sigma)} = E^{(\sigma)}(D^{(\sigma)}) \equiv \int_0^{D^{(\sigma)2}} \varepsilon \, \mathrm{d}D^{(\sigma)2} \qquad [47]$$

and

$$E_c^{(\sigma)} = E_c^{(\sigma)}(S^{(\sigma)}) \equiv \int_0^{S^{(\sigma)2}} \frac{1}{4\varepsilon} \, \mathrm{d}S^{(\sigma)2} \qquad [48]$$

such that

$$\mathbf{S}^{(\sigma)} = \frac{\partial E^{(\sigma)}}{\partial \mathbf{D}^{(\sigma)}} \qquad [49]$$

and

$$\mathbf{D}^{(\sigma)} = \frac{\partial E_c^{(\sigma)}}{\partial \mathbf{S}^{(\sigma)}}. \qquad [50]$$

In [48], we have defined

$$S^{(\sigma)} \equiv [\mathrm{tr}\,(\mathbf{S}^{(\sigma)} \cdot \mathbf{S}^{(\sigma)})]^{1/2}. \qquad [51]$$

These two potential functions are related by

$$E^{(\sigma)} + E_c^{(\sigma)} = S^{(\sigma)} D^{(\sigma)} = \mathrm{tr}\,(\mathbf{S}^{(\sigma)} \cdot \mathbf{D}^{(\sigma)}). \qquad [52]$$

Both of these potential functions can be required to be convex:

$$E^{(\sigma)}(D^{(\sigma)*}) - E^{(\sigma)}(D^{(\sigma)}) \geq \mathrm{tr}\left[\frac{\partial E^{(\sigma)}}{\partial \mathbf{D}^{(\sigma)}} \cdot (\mathbf{D}^{(\sigma)*} - \mathbf{D}^{(\sigma)})\right] \qquad [53]$$

$$E_c^{(\sigma)}(S^{(\sigma)*}) - E_c^{(\sigma)}(S^{(\sigma)}) \geq \mathrm{tr}\left[\frac{\partial E_c}{\partial \mathbf{S}^{(\sigma)}} \cdot (\mathbf{S}^{(\sigma)*} - \mathbf{S}^{(\sigma)})\right]. \qquad [54]$$

Sufficient conditions for their convexity are

$$2\varepsilon = \frac{1}{D^{(\sigma)}} \frac{\mathrm{d}E^{(\sigma)}}{\mathrm{d}D^{(\sigma)}} \geq 0 \qquad [55]$$

and

$$\frac{\mathrm{d}S^{(\sigma)}}{\mathrm{d}D^{(\sigma)}} = \frac{\mathrm{d}^2 E^{(\sigma)}}{\mathrm{d}D^{(\sigma)2}} \geq 0. \qquad [56]$$

These conditions are consistent with the limited observations of interfacial behavior available in the literature.

Recognizing [49], let us integrate [53] over the phase interface,

$$\int_\Sigma \{E^{(\sigma)}(D^{(\sigma)*}) - E(D^{(\sigma)}) - \mathrm{tr}\,[\mathbf{S}^{(\sigma)} \cdot (\mathbf{D}^{(\sigma)*} - \mathbf{D}^{(\sigma)})]\} \, \mathrm{d}A \geq 0. \qquad [57]$$

In this equation, we regard $\mathbf{D}^{(\sigma)*}$ as being defined in terms of the trial or approximate velocity distribution $\mathbf{v}^*$ introduced in the previous section. In view of [41] and [45], we may write

$$\begin{aligned}\operatorname{tr}(\mathbf{S}^{(\sigma)}\cdot\mathbf{D}^{(\sigma)}) &= \operatorname{tr}(\mathbf{T}^{(\sigma)}\cdot\mathbf{D}^{(\sigma)})\\ &= \operatorname{tr}(\mathbf{T}^{(\sigma)}\cdot\nabla_{(\sigma)}\mathbf{v}^{(\sigma)})\\ &= \operatorname{tr}(\mathbf{T}^{(\sigma)}\cdot\nabla_{(\sigma)}\mathbf{v}). \end{aligned} \quad [58]$$

By an application of Green's transformation for a surface (McConnell 1957), we are able to say

$$\int_\Sigma \operatorname{tr}(\mathbf{S}^{(\sigma)}\cdot\mathbf{D}^{(\sigma)})\,\mathrm{d}A = \oint_C \mathbf{v}\cdot\mathbf{T}^{(\sigma)}\cdot\boldsymbol{\mu}\,\mathrm{d}s - \int_\Sigma \mathbf{v}\cdot\operatorname{div}_{(\sigma)}\mathbf{T}^{(\sigma)}\,\mathrm{d}A. \quad [59]$$

Here C denotes the closed bounding curve of the phase interface Σ; $\boldsymbol{\mu}$ is the outwardly directed unit normal to the curve C; $\mathrm{d}s$ indicates that a line integration is to be performed. Using this result, we may express [57] as

$$\begin{aligned}\int_\Sigma E^{(\sigma)}(D^{(\sigma)})\,\mathrm{d}A \le &\int_\Sigma E^{(\sigma)}(D^{(\sigma)*})\,\mathrm{d}A\\ &+\oint_C (\mathbf{v}-\mathbf{v}^*)\cdot\mathbf{T}^{(\sigma)}\cdot\boldsymbol{\mu}\,\mathrm{d}s\\ &-\int_\Sigma (\mathbf{v}-\mathbf{v}^*)\cdot(\operatorname{div}_{(\sigma)}\mathbf{T}^{(\sigma)})\,\mathrm{d}A.\end{aligned} \quad [60]$$

Recognizing [50], we may now integrate [54] over the interface,

$$\int_\Sigma \{E_c^{(\sigma)}(S^{(\sigma)*}) - E_c^{(\sigma)}(S^{(\sigma)}) - \operatorname{tr}\,[\mathbf{D}^{(\sigma)}\cdot(\mathbf{S}^{(\sigma)*}-\mathbf{S}^{(\sigma)})]\}\,\mathrm{d}A \ge 0. \quad [61]$$

In view of [52], we may also express this as

$$\int_\Sigma \{E_c^{(\sigma)}(S^{(\sigma)*}) + E^{(\sigma)}(D^{(\sigma)}) - \operatorname{tr}(\mathbf{D}^{(\sigma)}\cdot\mathbf{S}^{(\sigma)*})\}\,\mathrm{d}A \ge 0. \quad [62]$$

Finally, by analogy with [59] we conclude

$$\begin{aligned}\int_\Sigma E^{(\sigma)}(D^{(\sigma)})\,\mathrm{d}A \ge &-\int_\Sigma E_c^{(\sigma)}(S^{(\sigma)*})\,\mathrm{d}A\\ &+\oint_C \mathbf{v}\cdot\mathbf{T}^{(\sigma)*}\cdot\boldsymbol{\mu}\,\mathrm{d}s\\ &-\int_\Sigma \mathbf{v}\cdot\operatorname{div}_{(\sigma)}\mathbf{T}^{(\sigma)*}\,\mathrm{d}A.\end{aligned} \quad [63]$$

Adding [30] and [60], we have

$$\int_R E(D)\,\mathrm{d}V + \int_\Sigma E^{(\sigma)}(D^{(\sigma)})\,\mathrm{d}A \le \int_R E(D^*)\,\mathrm{d}V + \int_\Sigma E^{(\sigma)}(D^{(\sigma)*})\,\mathrm{d}A + \int_{S-S_v} (\mathbf{v}-\mathbf{v}^*)\cdot(\mathbf{T}-\rho\varphi\mathbf{I})\cdot\mathbf{n}\,\mathrm{d}A + \oint_C (\mathbf{v}-\mathbf{v}^*)\cdot\mathbf{T}^{(\sigma)}\cdot\boldsymbol{\mu}\,\mathrm{d}s. \qquad [64]$$

In arriving at this result we have employed the jump momentum balance [40] and we have observed that, in view of assumption (8″), the normal components of both $\mathbf{v}$ and $\mathbf{v}^*$ are zero at Σ. Similarly, the addition of [33] and [63] yields

$$\int_R E(D)\,\mathrm{d}V + \int_\Sigma E^{(\sigma)}(D^{(\sigma)})\,\mathrm{d}A \ge -\int_R E_c(S^*)\,\mathrm{d}V - \int_\Sigma E_c^{(\sigma)}(S^{(\sigma)*})\,\mathrm{d}A + \int_S \mathbf{v}\cdot(\mathbf{T}^* - \rho^*\varphi\mathbf{I})\cdot\mathbf{n}\,\mathrm{d}A + \oint_C \mathbf{v}\cdot\mathbf{T}^{(\sigma)*}\cdot\boldsymbol{\mu}\,\mathrm{d}s. \qquad [65]$$

Here we recognized that the trial stress distributions $\mathbf{T}^*$ and $\mathbf{T}^{(\sigma)*}$ must be consistent with the jump momentum balance [40] at Σ and that the normal component of $\mathbf{v}$ must be zero at Σ.

To summarize, the trial velocity distribution $\mathbf{v}^{(\sigma)}$ and $\mathbf{v}^*$ must satisfy the jump mass balance [38], the equation of continuity, and all of the boundary conditions on velocity. The trial stress distributions $\mathbf{T}^*$ and $\mathbf{T}^{(\sigma)*}$ must be symmetric; they must also be consistent with the equation of motion [18] and the jump momentum balance [40]. We define $\mathbf{S}^*$ in terms of $\mathbf{T}^*$ by [32] and $\mathbf{S}^{(\sigma)*}$ in terms of $\mathbf{T}^{(\sigma)*}$ by

$$\mathbf{S}^{(\sigma)*} \equiv \mathbf{T}^{(\sigma)*} - \tfrac{1}{2}(\operatorname{tr}\mathbf{T}^{(\sigma)*})\mathbf{P}. \qquad [66]$$

We will restrict ourselves to potential functions $E^{(\sigma)}$ that are homogeneous of degree p (Kaplan 1952):

$$pE^{(\sigma)} = \operatorname{tr}\left(\frac{\partial E^{(\sigma)}}{\partial \mathbf{D}^{(\sigma)}}\cdot\mathbf{D}^{(\sigma)}\right) = \operatorname{tr}(\mathbf{S}^{(\sigma)}\cdot\mathbf{D}^{(\sigma)}). \qquad [67]$$

For the linear Boussinesq surface fluid model (Boussinesq 1913; Scriven 1960; Slattery 1964), ε is a constant and $E^{(\sigma)}$ is a homogeneous function of degree two:

$$E^{(\sigma)} = \varepsilon D^{(\sigma)2}. \qquad [68]$$

Another example is provided by the surface power model fluid for which

$$\varepsilon = \varepsilon_0 [2D^{(\sigma)2}]^{(n-1)/2}$$
$$= \varepsilon_0 \left[\frac{S^{(\sigma)2}}{2\varepsilon_0^2}\right]^{(n-1)/2n}. \qquad [69]$$

In writing this expression, we have recognized that

$$S^{(\sigma)2} = 2\varepsilon_0^2(2D^{(\sigma)2})^n. \qquad [70]$$

It follows immediately that for the surface power model fluid

$$E^{(\sigma)} = \frac{\varepsilon_0}{n+1}(2D^{(\sigma)2})^{(n+1)/2} \qquad [71]$$

$$E_c^{(\sigma)} = \frac{\varepsilon_0^n}{n+1}\left(\frac{S^{(\sigma)2}}{2\varepsilon_0^2}\right)^{(n+1)/2n}, \qquad [72]$$

and $E^{(\sigma)}$ is a homogeneous function of degree $n + 1$.

When we restrict ourselves to homogeneous potential functions $E^{(\sigma)}$ that are of the *same* degree r as E in [15], then

$$r\left\{\int_R E(D)\,\mathrm{d}V + \int_\Sigma E^{(\sigma)}(D^{(\sigma)})\,\mathrm{d}A\right\}$$
$$= \int_R \operatorname{tr}(\mathbf{S}\cdot\mathbf{D})\,\mathrm{d}V + \int_\Sigma \operatorname{tr}(\mathbf{S}^{(\sigma)}\cdot\mathbf{D}^{(\sigma)})\,\mathrm{d}A. \qquad [73]$$

As a consequence, [64] and [65] give us upper and lower bounds on the rate of dissipation of mechanical energy resulting from the action of viscous forces both in the bulk phase and in the phase interface.

More often we are concerned with Newtonian bulk fluid behavior, for which E is a homogeneous function of degree 2, and with nonlinear interfacial behavior. Let us assume that the interface may be described by a surface power model fluid, for which $E^{(\sigma)}$ is a homogeneous function of degree $n + 1$ and for which $n \leq 1$. In this case,

$$\int_R E(D)\,\mathrm{d}V + \int_\Sigma E^{(\sigma)}(D^{(\sigma)})\,\mathrm{d}A \geq \frac{1}{2}\left\{\int_R \operatorname{tr}(\mathbf{S}\cdot\mathbf{D})\,\mathrm{d}V + \int_\Sigma \operatorname{tr}(\mathbf{S}^{(\sigma)}\cdot\mathbf{D}^{(\sigma)})\,\mathrm{d}A\right\}, \qquad [74]$$

and

$$\int_R E(D)\,\mathrm{d}V + \int_\Sigma E^{(\sigma)}(D^{(\sigma)})\,\mathrm{d}A \leq \frac{1}{n+1}\left\{\int_R \operatorname{tr}(\mathbf{S}\cdot\mathbf{D})\,\mathrm{d}V + \int_\Sigma \operatorname{tr}(\mathbf{S}^{(\sigma)}\cdot\mathbf{D}^{(\sigma)})\,\mathrm{d}A\right\}. \qquad [75]$$

These two inequalities can in turn be combined with [64] and [65] to provide upper and lower bounds on the rate of viscous dissipation of mechanical energy.

As a simple application, consider a steady-state two-phase system with no entrances or exits and for which either [73] or [74] and [75] are applicable. From the integral mechanical energy balances (Slattery 1972), we see that [64] and [65] provide us with upper and lower bounds on

$$\mathscr{W} \equiv \int_{S_{(m)}} \mathbf{v} \cdot (\mathbf{T} + p_0 \mathbf{I}) \cdot (-\mathbf{n})\, \mathrm{d}A + \oint_C \mathbf{v} \cdot \mathbf{T}^{(\sigma)} \cdot (-\boldsymbol{\mu})\, \mathrm{d}s - \int_R \operatorname{tr}(\mathbf{S} \cdot \mathbf{D})\, \mathrm{d}V - \int_\Sigma \operatorname{tr}(\mathbf{S}^{(\sigma)} \cdot \mathbf{D}^{(\sigma)})\, \mathrm{d}A, \qquad [76]$$

the rate at which work is done by the system on the surroundings at the moving, impermeable surfaces $S_{(m)}$ of the system (beyond any work done on these surfaces by the ambient pressure).

SUMMARY

Bounding principles have been developed here for two classes of multiphase flow problems.

In the first class of problems, all interfacial effects other than a uniform surface tension must be neglected and the stress tensor for the bulk fluid must be described in terms of a potential function E that is homogeneous of degree r, as indicated in [15]. Further, some *a priori* knowledge of both the interface configuration and the velocity distribution in the neighborhood of the phase interface is required in order to justify the assumptions (8′) and (9′). When all of these conditions are met, [35] and [36] lead to upper and lower bounds on the rate of viscous dissipation of mechanical energy in the bulk phase.

In the second class of problems, we require *a priori* knowledge of both the configuration and location of the phase interface. If both E and $E^{(\sigma)}$ are homogeneous functions of the same degree r, then [64] and [65] give us upper and lower bounds on the viscous dissipation of mechanical energy. If E and $E^{(\sigma)}$ are homogeneous functions of different degrees, [64] and [65] lead to bounds on the viscous dissipation of mechanical energy when used with easily developed inequalities such as [74] and [75].

The rate of viscous dissipation of mechanical energy is related through the integral mechanical energy balance (Slattery 1972) to quantities that are subject to direct experimental observation.

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APPENDIX

Comments on notation

Some of the notation used in the text represents an extension to differential geometry of the coordinate-free notation of tensor analysis (Slattery 1972). The results are completely consistent with the presentation of McConnell (1957) including the use of the summation convention with repeated indices.

A surface is the locus of a point whose position vector $\mathbf{z}$ is a function of two parameters y^1 and y^2:

$$\mathbf{z} = \mathbf{p}^{(\sigma)}(y^1, y^2). \qquad [A1]$$

Since the two numbers y^1 and y^2 uniquely determine a point on the surface, we call them *surface coordinates*.

At every point on the surface, the values of the spatial vector fields

$$\mathbf{a}_\alpha \equiv \frac{\partial \mathbf{p}^{(\sigma)}}{\partial y^\alpha} = \frac{\partial x^i}{\partial y^\alpha}\mathbf{g}_i \qquad (\alpha = 1, 2) \qquad [A2]$$

are tangent to the y^α coordinate curves and therefore tangent to the surface. These two vectors are linearly independent and every vector tangent to the surface can be written as a linear combination of them. We can refer to them as the *natural basis* fields.

Let us define

$$a_{\alpha\beta} \equiv \mathbf{a}_\alpha \cdot \mathbf{a}_\beta. \qquad [A3]$$

These are often referred to as the covariant components of the metric tensor (McConnell 1957). Let a be the determinant whose elements are the $a_{\alpha\beta}$ and let $a^{\alpha\beta}$ be the cofactor of $a_{\alpha\beta}$ in a, divided by a. The *dual basis* fields

$$\mathbf{a}^\alpha = a^{\alpha\beta}\mathbf{a}_\beta \qquad (\alpha = 1, 2) \qquad [A4]$$

are also linearly independent. Every vector tangent to the surface can be written as a linear combination of them as well.

The surface velocity vector $\mathbf{v}^{(\sigma)}$ is an example of a vector defined on the dividing surface that has normal and tangential components:

$$\begin{aligned}\mathbf{v}^{(\sigma)} &= v^{(\sigma)\alpha}\mathbf{a}_\alpha + (\mathbf{v}^{(\sigma)} \cdot \boldsymbol{\xi})\boldsymbol{\xi} \\ &= v^{(\sigma)}_\alpha \mathbf{a}^\alpha + (\mathbf{v}^{(\sigma)} \cdot \boldsymbol{\xi})\boldsymbol{\xi}.\end{aligned} \quad [A5]$$

The $v^{(\sigma)\alpha}$ are said to be the *contravariant tangential* components of $\mathbf{v}^{(\sigma)}$; the $v^{(\sigma)}_\alpha$ are the *covariant tangential* components.

The *projection* tensor

$$\mathbf{P} = \mathbf{a}_\alpha \mathbf{a}^\alpha = \mathbf{a}^\alpha \mathbf{a}_\alpha \quad [A6]$$

transforms every vector on the surface into its tangential component.

The *surface gradient* of a scalar such as surface tension is defined as

$$\nabla_{(\sigma)}\gamma = \frac{\partial \gamma}{\partial y^\alpha}\mathbf{a}^\alpha. \quad [A7]$$

A vector such as surface velocity $\mathbf{v}^{(\sigma)}$ is an explicit function of position on the dividing surface. The *surface gradient* of such a vector is similarly

$$\nabla_{(\sigma)}\mathbf{v}^{(\sigma)} = \frac{\partial \mathbf{v}^{(\sigma)}}{\partial y^\alpha}\mathbf{a}^\alpha. \quad [A8]$$

We have a particular interest in the surface rate of deformation tensor, which becomes

$$\begin{aligned}\mathbf{D}^{(\sigma)} &\equiv \tfrac{1}{2}[\mathbf{P} \cdot \nabla_{(\sigma)}\mathbf{v}^{(\sigma)} + (\nabla_{(\sigma)}\mathbf{v}^{(\sigma)})^T \cdot \mathbf{P}] \\ &= \tfrac{1}{2}(v^{(\sigma)}_{\alpha,\beta} + v^{(\alpha)}_{\beta,\alpha})\mathbf{a}^\alpha \mathbf{a}^\beta,\end{aligned} \quad [A9]$$

where the comma denotes surface covariant differentiation (McConnell 1957). Finally, the *surface divergence* of a vector such as $\mathbf{v}^{(\sigma)}$ is

$$\mathrm{div}_{(\sigma)}\,\mathbf{v}^{(\sigma)} \equiv \mathrm{tr}\,(\nabla_{(\sigma)}\mathbf{v}^{(\sigma)}) = v^{(\sigma)\alpha}_{,\alpha} - 2H\mathbf{v}^{(\sigma)} \cdot \boldsymbol{\xi}. \quad [A10]$$

The symmetric surface tensor $\mathbf{T}^{(\sigma)}$ is a type of tensor defined on the dividing surface that transforms tangential vectors into tangential vectors and normal vectors into the zero vector:

$$\begin{aligned}\mathbf{T}^{(\sigma)} &= \mathbf{P} \cdot \mathbf{T}^{(\sigma)} \cdot \mathbf{P} \\ &= T^{(\sigma)\alpha\beta}\mathbf{a}_\alpha \mathbf{a}_\beta.\end{aligned} \quad [A11]$$

We can define the surface gradient of such a tensor in a manner very similar to our definitions for the surface gradient of a scalar and the surface gradient of a vector. Of particular interest to us here is the *surface divergence* of this type of tensor:

$$\begin{aligned}\mathrm{div}_{(\sigma)}\,\mathbf{T}^{(\sigma)} &= \left(\frac{\partial x^i}{\partial y^\alpha}T^{\alpha\beta}\right)_{,\beta}\mathbf{j}_i \\ &= T^{\alpha\beta}_{,\beta}\mathbf{a}_\alpha + T^{\alpha\beta}B_{\alpha\beta}\boldsymbol{\xi}.\end{aligned} \quad [A12]$$

The comma again denotes surface covariant differentiation (McConnell 1957). The $B_{\alpha\beta}$ are the components of the second groundform tensor (McConnell 1957).

Résumé—Pour deux classes de problèmes d'écoulements multiphasiques, on construit des bornes supérieure et inférieure du taux de dissipation d'énergie mécanique résultant des forces visqueuses s'exerçant à la fous dans le fluide proprement dit et aux interfaces eutre phases. On développe ces principes pour des classes simples d'équations constitutives non-lineáires pour les tenseurs de contraintes de volume et de surface. Le bilau global d'énergie mécanique relie ces bornes à des quantités susceptibles d'une évaluation expérimentale directe.

Auszug—Fuer zwei Klassen von Problemen der Mehrphasenstroemung werden obere und untere Grenzen fuer die Dissipationsgeschwindigkeit mechanischer Energie festgelegt, die eine Folge der Zaehigkeitskraefte sowohl in der Hauptstroemung wie auch in der Phasentrennschicht ist. Diese Prinzipien werden fuer einfache Klassen von nichtlinearen Grundgleichungen entwickelt, fuer die Spannungstensoren in der Hauptstroemung und an der Oberflaeche. Die Integralbilanz der mechanischen Energie setzt diese Grenzen zu Groessen in Beziehung, die direkter experimenteller Nachpruefung zugaenglich sind.

Резюме—Устанавливаются верхняя и нижняя границы для двух классов задачи многофаз; ного течения с чуетом скорости рассеяния механической энергии вследствие действия сил вязкости как объеме жидкости, так и на межфазной границе. Указанные принципы развиты для простых классов нелинейных основных (канонических) уравнений, относящихся к тензорам объемионапряженного и поверхностнонапряженного состояний. Разность интегральной механической энергии устанавливает связь этих граничных условий с количествами, подлежащими непосредственной экспериментальной оценке.